CR-Submanifolds of an (ϵ)-Lorentzian Para-Sasakian Manifold Endowed with Quarter Symmetric Non-Metric Connection

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Abstract- In this paper we study quarter-symmetric non metric connection in CR-Submanifolds of (ϵ)- LP-Sasakian manifold. Some results related to this connection are obtained and studied. Also we dealt with totally geodesic and umbilic.

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Key words- (ϵ)-Lorentzian Para-sasakian manifold, quarter symmetric, non metric, distribution geodesic, CR-Submanifolds, umbilic.

1 INTRODUCTION

CR-submanifolds were introduced first in kaehler geometry. It works as a bridge between complex and totally real submanifolds. In 1978, A. Bejancu introduced the notion of CR-submanifolds of the kaehler manifold [1, 2]. After that CR-submanifolds of Sasakian manifold were studied by M.Kobayashi in [7]. In 1989 K.Motsumoto introduced the notion of the Lorentzian para-Sasakian manifold [5]. I.Mihai and R.Rosca [4] defined the same idea independently and several others authors were studied Lorentzian para-Sasakian manifold (briefly LP-Sasakian Manifold).

In [3] Bejancu and Duggal introduced (ϵ)-Sasakian manifolds. Later, Xufeng and Xiaoli [14] showed that every (ϵ)-Sasakian manifold must be real hypersurface of some indefinite Kahler manifold. In 2009, U.C. De and A. sarkar [13] give the idea of (ϵ)-Kenmotsu manifolds. Recently, in 2012 R. Prasad and V. srivastva [8] introduced the (ϵ)-Lorentzian Para-sasakian manifold.

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In this paper we study CR-submanifolds of (ϵ)-Lorentzian Para-sasakian manifold endowed with quarter symmetric non-metric connection which include the usual LP-sasakian manifold. Let ∇ be a linear connection in n dimensional differentiable manifold M. The Torsion tensor is defined as

$$(1.1)\quad T(X,Y)=\nabla_X Y-\nabla_Y X-[X,Y]$$

And the curvature tensor R is defined as

$$(1.2)\quad R(X,Y,Z)=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z$$

If the torsion tensor T vanishes then the connection ∇ is symmetric otherwise non symmetric. Again if $\nabla_g=0$, then ∇ is metric connection otherwise it is non metric connection, where g is Riemannian metric in M. S.Golab[9] introduced the idea of a quarter symmetric connection. A linear connection is said to be quarter symmetric connection if its torsion tensor is the form of

$$(1.3)\quad T(X,Y)=\eta(Y)\phi X-\eta(X)\phi Y$$

Where η is 1-form. This was further developed by Yano and Kon [6], Rastogi [10], Mishra and Pandey [9], Mukhopadhyay, Roy and Barua [12] and many others authors.

This paper is organized as follows:

In section 2, we give the brief introduction of (ϵ)-Lorentzian Para-sasakian manifold. In section 3 we prove some basic lemmas on (ϵ)-Lorentzian Para-sasakian manifold. We discuss the parallel distribution in section 4. At last in section 5 we prove some results base on totally geodesic and umbilic.

2. PRELIMINARIES

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An n dimensional differentiable manifold $\bar{M}$ is called (ε)-Lorentzian para-Sasakian manifold if:

(2.1) $\phi^2 = I + \eta(X)\xi$, $\eta(\xi) = -1$, $\phi \circ \xi = 0$

(2.2) $g(\xi,\xi) = \varepsilon$, $\eta(X) = \varepsilon g(X,\xi)$

(2.3) $g(\phi X, \phi Y) = g(X,Y) + \varepsilon\eta(X)\eta(Y)$

Where X and Y are the vector fields tangent to $\bar{M}$ and ε is 1 or -1 according as ξ is space like or time like vector field.

Also in (ε)-Lorentzian para-Sasakian manifold, we have

(2.4) $(\bar{\bar{\nabla}}_X \phi)Y = g(X,Y)\xi + \varepsilon\eta(Y)X + 2\varepsilon\eta(X)\eta(Y)$

where $\bar{\bar{\nabla}}$ denotes the operator of covariant differetiation with respect to the Lorentzian metric g on $\bar{M}$.

(2.5) $\bar{\nabla}_X \xi = \varepsilon\phi X$

(2.6) $\Phi(X,Y) = g(X,\phi Y)$

(2.7) $g(\phi X, Y) = (\bar{\bar{\nabla}}_X \eta)Y$

Where $\Phi(X,Y)$ is symmetric $(0,2)$ tensor field.

Now, we remark the owning the existence of 1-form η, we can define the quarter symmetric non metric connection $\bar{\nabla}$ by

(2.8) $\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y + \varepsilon\eta(Y)\phi X + a(X)\phi Y$

Such that

(2.9) $(\bar{\nabla}_X g)(Y,Z) = -\eta(Y)g(\phi X, Z) - \eta(Z)g(\phi X, Y) - 2\alpha(X)\phi Y$

For any $X, Y \in T\bar{M}$ and ξ is vector field.

Using (2.4) and (2.8), we get

(2.10) $(\bar{\nabla}_X \phi)Y = g(X,Y)\xi - \varepsilon\eta(X)\eta(Y)\xi + 2\varepsilon\eta(X)\eta(Y)$

This implies

(2.11) $(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = 2g(X,Y)\xi - 2\varepsilon\eta(X)\eta(Y)\xi + 4\varepsilon\eta(X)\eta(Y)$

From (2.5) and (2.8), we get

(2.12) $\bar{\nabla}_X \xi = 0$

Definition : An m dimensional Riemannian submanifold M of (ε)-Lorentzian para-Sasakian manifold $\bar{M}$ is called a CR-Submanifold if ξ is tangent to M and there exists a differentiable distribution $D: x \in M \to D_x \subset T_x M$ such that

(i) The distribution D_x is invariant under ϕ, that is

$\phi D_X \subset D_X$ for each $x \in M$;

(ii) The complementary orthogonal distribution $D^\perp : x \in M \to D_X^\perp \subset T_X M$ of D is anti-invariant under ϕ that is

$\phi D_X^\perp \subset T_X^\perp M$ for each $x \in M$;

Where $T_x M$ and $T_X^\perp M$ are the tangent space and the normal space of M at x respectivly.

If $\dim D_x^\perp = 0$ (resp., dim D_x= 0), then the CR-Submanifold is called an invariant (resp., anti-invariant) submanifold. The distribution D (resp., $D^\perp$) is called the horizontal (resp., vertical) distribution.

Also the pair $(D, D^\perp)$ is called ξ –horizontal (resp., vertical) if $\xi_x \in D_x$ (resp., $\xi_X \in D^\perp$)[10].

For any vector field X tangent to M, we put [10]

(2.13) $X = PX + QX$

For any vector field normal to M, we have

(2.14) $\phi N = BN + CN$

Where BN and CN denote the tangential and normal component of ϕN respectively.

Let $\bar{\nabla}$ (resp., ∇) be the covariant differentiation with respect to the Levi-civita connection on $\bar{M}$(resp., M). The Gauss and Weingarten formulas for M are respectively given by

(2.15) $$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

(2.16) $$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N$$

For X, Y ∈ TM and $N \in T^{\perp}M$ where h (resp., A) is second fundamental form (resp., tensor) of M in $\overline{M}$ and $\nabla^{\perp}$ denotes the normal connection. Moreover, we have

(2.17) $$g(h(X,Y),N) = g(A_N X,Y)$$

3. SOME BASIC LEMMAS

Lemma 3.1: Let M be a CR-submanifold of an (ε)-Lorentzian para-Sasakian manifold $\overline{M}$ with quarter symmetric non metric connection. Then

(3.1) $$P(\nabla_X \phi PY) + P(\nabla_Y \phi PX) - P(A_{\phi QY} X) - P(A_{\phi QX} Y) = 2g(X,Y)P\xi - 2\varepsilon\eta(X)\eta(Y)P\xi + \phi(P\nabla_X Y) + \phi(P\nabla_Y X) + 4\varepsilon\eta(X)\eta(Y)$$

(3.2) $$Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) - Q(A_{\phi QY} X) - Q(A_{\phi QX} Y) = 2Bh(X,Y) + 2g(X,Y)Q\xi - 2\varepsilon\eta(X)\eta(Y)Q\xi$$

(3.3) $$h(X,\phi PY) + h(Y,\phi PX) + \nabla_X^{\perp}\phi QY + \nabla_Y^{\perp}\phi QX = 2Ch(X,Y) + \phi Q\nabla_X Y + \phi Q\nabla_Y X$$

For all $X,Y \in TM$.

Proof. By the virtue of (2.11), (2.13), (2.14), (2.15) and (2.16) we get

(3.4) $$\nabla_X \phi PY + \nabla_Y \phi PX + h(X,\phi PY) + h(Y,\phi PX) - A_{\phi QY} X - A_{\phi QX} Y + \nabla_X^{\perp}\phi QY + \nabla_Y^{\perp}\phi QX = 2g(X,Y)\xi - 2\varepsilon\eta(X)\eta(Y)\xi + \phi(\nabla_X Y) + \phi(\nabla_Y X) + 4\varepsilon\eta(X)\eta(Y)$$

After equating the horizontal, vertical and normal components of the above equation (3.4) we get the lemma.

Lemma 3.2: Let M be a CR-submanifold of an (ε)-Lorentzian para-Sasakian manifold $\overline{M}$ with a quarter symmetric non metric connection. Then

(3.5) $$2(\overline{\nabla}_X \phi)Y = \nabla_X \phi Y - \nabla_Y \phi X + h(X,\phi Y) - h(Y,\phi X) + 2g(X,Y)\xi - 2\varepsilon\eta(X)\eta(Y)\xi + 4\varepsilon\eta(X)\eta(Y) - \phi[X,Y]$$

For any $X,Y \in D$

Proof. By using the Gauss formula (2.15), we get

(3.6) $$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = \nabla_X \phi Y - \nabla_Y \phi + h(X,\phi Y) - h(Y,\phi X)$$

Also we know

(3.7) $$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = \nabla_X \phi Y - \nabla_Y \phi X + \phi[X,Y]$$

From (3.6) and (3.7), we get

(3.8) $$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = \nabla_X \phi Y - \nabla_Y \phi + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y]$$

Again from (2.11) and (3.8), we get the result.

Lemma 3.3: Let M be a CR-submanifold of an (ε)-Lorentzian para-Sasakian manifold $\overline{M}$ with a quarter symmetric non metric connection. Then

(3.9) $$2(\overline{\nabla}_Y \phi)Z = 2g(Y,Z)\xi - 2\varepsilon\eta(Y)\eta(Z)\xi + 4\varepsilon\eta(Y)\eta(Z) - A_{\phi Z} Y + \nabla_Y^{\perp}\phi Z - A_{\phi Y} Z + \nabla_Z^{\perp}\phi Y - \phi[Y,Z]$$

For any $Y,Z \in D^{\perp}$.

Proof. From Weingarten formula (2.16), we get

(3.10) $$\overline{\nabla}_Y \phi Z - \overline{\nabla}_Z \phi Y = -A_{\phi Z} Y + A_{\phi Y} Z + \nabla_Y^{\perp}\phi Z - \nabla_Z^{\perp}\phi Y$$

Also we know that

(3.11) $$\overline{\nabla}_Y \phi Z - \overline{\nabla}_Z \phi Y = (\overline{\nabla}_Y \phi)Z - (\overline{\nabla}_Z \phi)Y + \phi[Y,Z]$$

Now from (3.10) and (3.11), we get

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(3.12) $$\left(\overline{\nabla}_Y\phi\right)Z-\left(\overline{\nabla}_Z\phi\right)Y=-A_{\phi Z}Y+A_{\phi Y}Z+\nabla_Y^{\perp}\phi Z-\nabla_Z^{\perp}\phi Y-\phi[Y,Z]$$

Also for an (ε)-Lorentzian para-Sasakian manifold $\overline{M}$ with a quarter symmetric non metric connection, we have

(3.13) $$\left(\overline{\nabla}_Y\phi\right)Z+\left(\overline{\nabla}_Z\phi\right)Y=2g(Y,Z)\xi-2\varepsilon\eta(Y)\eta(Z)\xi+4\varepsilon\eta(Y)\eta(Z)$$

By adding (3.12) and (3.13), we have the lemma.

Lemma 3.4: Let M be a CR-submanifold of an (ε)-Lorentzian para-Sasakian manifold $\overline{M}$ with a quarter symmetric non metric connection. Then

(3.14) $$2\left(\overline{\nabla}_X\phi\right)Y=2g(X,Y)\xi-2\varepsilon\eta(X)\eta(Y)\xi+4\varepsilon\eta(X)\eta(Y)-A_{\phi Y}X+\nabla_X^{\perp}\phi Y-\nabla_Y\phi X-h(Y,\phi X)-\phi[X,Y]$$

For $X\in D$, and $Y\in D^{\perp}$.

Proof. By using the Gauss (2.15) and Weingarten (2.16) formula, we get

(3.15) $$\overline{\nabla}_X\phi Y-\overline{\nabla}_Y\phi X=-A_{\phi Y}X+\nabla_X^{\perp}\phi Y-\nabla_Y\phi X-h(Y,\phi X)$$

And

(3.16) $$\left(\overline{\nabla}_X\phi\right)Y-\left(\overline{\nabla}_Y\phi\right)X=-A_{\phi Y}X+\nabla_X^{\perp}\phi Y-\nabla_Y\phi X-h(Y,\phi X)-\phi[X,Y]$$

By adding the equations (2.11) and (3.16), we have the lemma.

4. PARALLEL DISTRIBUTION

Definition : The horizontal (resp. vertical) distribution D (resp. $D^{\perp}$) is said to be parallel with respect to the quarter symmetric non metric connection on M if $\nabla_X\phi Y\in D$ (resp. $\nabla_W\phi Z\in D^{\perp}$) for any vector field X,Y∈D (resp. $W,Z\in D^{\perp}$).

Theorem 4.1: Let M be a ξ-vertical CR-submanifold of an (ε)-Lorentzian para-Sasakian manifold $\overline{M}$ with the quarter symmetric non metric connection. If the horizontal distribution D is parallel, then

(4.1) $$h(\phi X,Y)=h(X,\phi Y)$$

Proof. For horizontal distribution D we have, $\nabla_X\phi Y\in D$, $\nabla_Y\phi X\in D$ for any $X,Y\in D$

Using the fact QX = QY = 0, for any $X,Y\in D$,

(2.18) gives,

(4.2) $$Bh(X,Y)=\varepsilon\eta(X)\eta(Y)Q\xi-g(X,Y)Q\xi$$

(4.3) $$\phi h(X,Y)=Bh(X,Y)+Ch(X,Y)$$

(4.4) $$\phi h(X,Y)=\varepsilon\eta(X)\eta(Y)Q\xi-g(X,Y)Q\xi+Ch(X,Y)$$

From (3.2), we have

(4.5) $$h(X,\phi Y)+h(Y,\phi X)=2\phi h(X,Y)+2g(X,Y)Q\xi-2\varepsilon\eta(X)\eta(Y)Q\xi$$

By putting X= ϕX in equation (4.5), we get

(4.6) $$h(\phi X,\phi Y)+h(Y,X)=2\phi h(\phi X,Y)+2g(\phi X,Y)Q\xi$$

Again replace Y=ϕY in equation (4.5), we get

(4.7) $$h(X,Y)+h(\phi Y,\phi X)=2\phi h(X,\phi Y)+2g(X,\phi Y)Q\xi$$

Hence from (4.6) and (4.7), we get

(4.8) $$\phi h(\phi X,Y)-\phi h(X,\phi Y)+2g(\phi X,Y)Q\xi=0$$

Operating ϕ on both side, we have the theorem.

Now for $D^{\perp}$ we prove the following theorem:

Theorem 4.2: Let M be a ξ-vertical CR-submanifold of an (ε)-Lorentzian para-Sasakian manifold $\overline{M}$ quarter symmetric non metric connection. If the distribution $D^{\perp}$ is parallel with respect to the connection on M, then

(4.9) $$\left(A_{\phi Z}Y - A_{\phi Y}Z\right) \in D^{\perp}$$

Proof. Using the Weingarten formula (2.16), we get

(4.10) $$\overline{\nabla}_Z \phi Y - \overline{\nabla}_Y \phi Z = -A_{\phi Y}Z + A_{\phi Z}Y + \nabla_Z^{\perp}\phi Y - \nabla_Y^{\perp}\phi Z$$

We know that

(4.11) $$\overline{\nabla}_Z \phi Y - \overline{\nabla}_Y \phi Z = \left(\overline{\nabla}_Z \phi\right)Y - \left(\overline{\nabla}_Y \phi\right)Z - \phi\left(\overline{\nabla}_Y Z\right) + \phi\left(\overline{\nabla}_Z Y\right)$$

From (4.10) and (4.11), we get

(4.12) $$\left(\overline{\nabla}_Z \phi\right)Y - \left(\overline{\nabla}_Y \phi\right)Z - \phi\left(\overline{\nabla}_Y Z\right) + \phi\left(\overline{\nabla}_Z Y\right) = -A_{\phi Y}Z + A_{\phi Z}Y + \nabla_Z^{\perp}\phi Y - \nabla_Y^{\perp}\phi Z$$

Taking the inner product in (4.12) with respect to X∈D, We have

(4.13) $$g\left(-A_{\phi Y}Z, X\right) + g\left(A_{\phi Z}Y, X\right) = 0$$

Which implies

(4.14) $$g\left(A_{\phi Z}Y - A_{\phi Y}Z, X\right) = 0$$

Which gives the result.

5. TOTALLY GEODESIC

Definition : A CR-submanifold is said to be D-totally geodesic (resp. $D^{\perp}$-totally geodesic) if $h(X, Z) = 0$ for all $X, Y \in D$ (resp., $h(X, Z) = 0$, for all $X, Y \in D^{\perp}$).

Theorem 5.1: Let M be a CR-submanifold of an (ε)-Lorentzian para-Sasakian manifold $\overline{M}$ with a quarter symmetric non metric connection.

(i) Let M is D-totally geodesic if and only if $A_N X \in D^{\perp}$

(ii) Let M is $D^{\perp}$-totally geodesic if and only if $A_N X \in D$

Proof. From (2.17) and hypothesis (i), We have

(5.1) $$g\left(A_N X, Y\right) = 0 = g\left(h\left(X, N\right), Y\right)$$

This implies $h\left(X, Y\right) = 0$

Hence M is totally geodesic.

Similarly we prove the (ii).

Definition : A CR-submanifold with quarter symmetric non metric connection is said to be mixed totally geodesic if h(X,Z)=0 for all X belongs to horizontal distribution D and Y belongs to vertical distribution .

Lemma 5.2: Let M be a CR-submanifold of (**ε**)-Lorentzian para-Sasakian manifold M with the quarter symmetric non metric connection. Then M is mixed totally geodesic if and only if

$$A_N X \in D.$$

Lastly we study the umbilic,

Definition : A CR-Submanifold of (**ε**)-Lorentzian Para-Sasakian manifold M with quarter symmetric non metric connection is called D-umbilic (recp. $D^{\perp}$-umbilic) if

(5.2) h(X,Y)=g(X,Y)H,

for all X,Y $\in D$ $\left(resp. X, Y \in D^{\perp}\right)$ where H is mean curvature vector field.

Lemma 5.3: Let M be a D-umbilic ξ-horizontal CR-submaifold of a (ε)-Lorentzian Para-Sasakian manifold M with the quarter symmetric non metric connection, then M is D-totally geodesic.

Proof. Let M be D-umbilic ξ-horizontal CR-Submanifold with the quarter symmetric non metric connection, then by putting X=Y=ξ in (5.2),we get

(5.3) H=0

now using (5.3) , we have

h(X,Y)=0

which proves that M is D-totally geodesic.

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